

# On a problem of J. Nakagawa, K. Sakamoto, M. Yamamoto

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## Abstract

In this paper, we give a positive answer to a problem posed by Nakagawa, Sakamoto and Yamamoto concerning a nonlinear equation with a fractional derivative.

**Keywords:** Fractional differential equation, global existence, asymptotic behavior, blow-up time, blow-up profile.

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## 1. Introduction

In their overview paper concerning the mathematical analysis of fractional equations, Nakagawa, Sakamoto and Yamamoto [1] posed the problem concerning global solutions and blowing-up in a finite time of solutions to the equation

$$\begin{cases} {}^C D_{0+}^\alpha u(t) &= -u(t)(1-u(t)), \quad t > 0, \\ u(0) &= u_0, \end{cases} \quad (1)$$

where  ${}^C D_{0+}^\alpha$  is the Caputo derivative defined for  $g \in C^1[0, T]$  by

$${}^C D_{0+}^\alpha g(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} g'(\tau) d\tau,$$

for  $0 < \alpha < 1$ .

Let us recall, in the case  $\alpha = 1$ , the results concerning solutions of (1) :

- For  $0 < u(0) < 1$ , the solution exists globally. Moreover,

$$|u(t)| \leq \frac{1}{e^t(1-u_0)} \longrightarrow 0, \text{ as } t \longrightarrow +\infty.$$

- For  $u(0) > 1$ , the solution can not exist globally.

Here, we show that the same conclusions are valid for equation (1). Moreover we analyse :

1. The large time behavior of the global solution.
2. The blow-up time and profile of the blowing-up solutions.

Note that if we set  $w = u - 1$ , then (1) reads

$${}^C D_{0+}^\alpha w(t) = w(t)(1+w(t)),$$

which describes the evolution of a certain species; the reaction term  $w(1+w)$  describes the law of increase of the species.

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## 2. Preliminaries

In this section, we present some definitions and results concerning fractional calculus that will be used in the sequel. For more information see [2].

The Riemann-Liouville fractional integral of order  $0 < \alpha < 1$  of the integrable function  $f : \mathbb{R}^+ \rightarrow \mathbb{R}$  is

$$J_{0+}^{\alpha} f(t) := \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha-1} f(\tau) d\tau, \quad t > 0,$$

where  $\Gamma(\alpha)$  is the Euler Gamma function.

The Riemann-Liouville fractional derivative of an absolutely continuous function  $f(t)$  of order  $0 < \alpha < 1$  is

$$D_{0+}^{\alpha} f(t) := \frac{1}{\Gamma(1 - \alpha)} \frac{d}{dt} \int_0^t (t - \tau)^{-\alpha} f(\tau) d\tau.$$

The Caputo fractional derivative of an absolutely continuous function  $f(t)$  of order  $0 < \alpha < 1$  is defined by

$${}^C D_{0+}^{\alpha} f(t) := J_{0+}^{1-\alpha} f'(t) = \frac{1}{\Gamma(1 - \alpha)} \int_0^t (t - \tau)^{-\alpha} f'(\tau) d\tau.$$

Both derivatives present a drawback :

- The Riemann-Liouville derivative of a constant is different from zero,

$$D_{0+}^{\alpha} C \neq 0,$$

while the Caputo derivative require  $f'(t)$  to calculate  ${}^C D_{0+}^{\alpha} f(t)$ , for  $0 < \alpha < 1$ .

- We know that the Riemann-Liouville derivative of the Weierstrass function exists for any  $0 < \alpha < 1$ , but not for  $\alpha = 1$ .

But for regular function with  $f(0) = 0$ , both definitions coincide.

Next, we recall a lemma that will be used hereafter.

**Lemma 2.1.** ( see [3]). Let  $a, b, K, \psi$  be non-negative continuous functions on the interval  $I = (0, T)$  ( $0 < T \leq \infty$ ), let  $\omega : (0, \infty) \rightarrow \mathbb{R}$  be a continuous, non-negative and non-decreasing function with  $\omega(0) = 0$  and  $\omega(u) > 0$  for  $u > 0$ , and let  $A(t) = \max_{0 \leq s \leq t} a(s)$  and  $B(t) = \max_{0 \leq s \leq t} b(s)$ . Assume that

$$\psi(t) \leq a(t) + b(t) \int_0^t K(s) \omega(\psi(s)) ds, \quad t \in I.$$

Then

$$\psi(t) \leq H^{-1} \left[ H(A(t)) + B(t) \int_0^t K(s) ds \right], \quad t \in (0, T_1),$$

where  $H(v) = \int_{v_0}^v \frac{d\tau}{\omega(\tau)}$  ( $v \geq v_0 > 0$ ),  $H^{-1}$  is the inverse of  $H$  and  $T_1 > 0$  is such that  $H(A(t)) + B(t) \int_0^t K(s) ds \in D(H^{-1})$  for all  $t \in (0, T_1)$ .

Here, we consider the problem

$$\begin{cases} {}^C D_{0+}^{\alpha} u(t) &= -u(t)(1 - u(t)), \\ u(0) &= u_0, \end{cases} \quad (2)$$

for  $0 < \alpha < 1$  and  $u_0 > 0$ .

### 3. Main results

The local existence of solutions to (2) is assured by the

**Theorem 3.1.** (see [2]). We consider the fractional differential equation of Caputo's type given by

$$\begin{cases} {}^C D_{0+}^\alpha u(t) &= f(t, u(t)), \quad t > 0, \\ u(0) &= u_0. \end{cases} \quad (3)$$

For  $0 < \alpha < 1$ ,  $u_0 \in \mathbb{R}$ ,  $b > 0$  and  $T > 0$ .

Assume that

1.  $f \in C(R_0, \mathbb{R})$  where  $R_0 = \{(t, u), \quad 0 \leq t \leq T, \quad |u - u_0| \leq b\}$  and  $|f(t, u)| \leq M$  on  $R_0$ ;
2.  $|f(t, u) - f(t, v)| \leq L|u - v|$ ,  $L > 0$ ,  $(t, u) \in R_0$ .

Then there exists a unique solution  $u \in C([0, h])$  for (3), where  $h = \min\left\{T, \left(\frac{b\Gamma(\alpha+1)}{M}\right)^{\frac{1}{\alpha}}\right\}$ .

**Theorem 3.2.** Let  $u$  be the solution of problem (2). We have :

- If  $0 < u_0 < 1$ , the solution is global and it satisfies  $0 < u < 1$ . Moreover,  $u$  is given by

$$u(t) = E_\alpha(-t^\alpha)u_0 + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-(t-s)^\alpha) u^2(s) ds,$$

and for some constants  $c > 0$  and  $c_1 > 0$ , we have

$$0 < u(t) \leq \frac{1}{\frac{1}{cu_0} - \frac{c_1}{\alpha} t^\alpha}, \quad t > T_0 := \left(\frac{\alpha}{c_1 cu_0}\right)^{\frac{1}{\alpha}}.$$

- If  $u_0 > 1$ , the solution blows-up in a finite time  $T^* : \lim_{t \rightarrow T^*} u(t) = +\infty$ .

Moreover, we have the bilateral estimate :

$$\overline{w}(t) + 1 \leq u(t) \leq \widetilde{w}(t) + 1,$$

and

$$\left(\frac{\Gamma(\alpha+1)}{4(u_0 - \frac{1}{2})}\right)^{\frac{1}{\alpha}} \leq T^* \leq \left(\frac{\Gamma(\alpha+1)}{u_0 - 1}\right)^{\frac{1}{\alpha}},$$

where

$$\widetilde{w}(t) + \frac{1}{2} \sim \frac{\Gamma(2\alpha)}{\Gamma(\alpha)} (T_{\widetilde{w}} - t)^{-\alpha}, \quad \text{as } t \rightarrow T_{\widetilde{w}},$$

$$\overline{w}(t) \sim \frac{\Gamma(2\alpha)}{\Gamma(\alpha)} (T_{\overline{w}} - t)^{-\alpha}, \quad \text{as } t \rightarrow T_{\overline{w}}.$$

Here,  $T_{\widetilde{w}}$  is the blow-up time of  $\widetilde{w}$ , which satisfies

$$\left(\frac{\Gamma(\alpha+1)}{4(u_0 - \frac{1}{2})}\right)^{\frac{1}{\alpha}} \leq T_{\widetilde{w}} \leq \left(\frac{\Gamma(\alpha+1)}{u_0 - \frac{1}{2}}\right)^{\frac{1}{\alpha}},$$

and  $T_{\overline{w}}$  is the blow-up time of  $\overline{w}$ , which satisfies

$$\left(\frac{\Gamma(\alpha+1)}{4(u_0 - 1)}\right)^{\frac{1}{\alpha}} \leq T_{\overline{w}} \leq \left(\frac{\Gamma(\alpha+1)}{u_0 - 1}\right)^{\frac{1}{\alpha}}.$$

**Proof of Theorem 3.2.**

**Part 1.** If  $0 < u_0 < 1$ , then the solution is global.  
The solution to (2) is given by

$$u(t) = E_\alpha(-t^\alpha)u_0 + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-(t-s)^\alpha) u^2(s) ds. \quad (4)$$

Where the Mittag-Leffler functions  $E_\alpha(-t^\alpha)$  and  $E_{\alpha,\alpha}(-t^\alpha)$  are defined by :

$$E_\alpha(-t^\alpha) = \sum_{j=0}^{\infty} \frac{(-1)^j t^{\alpha j}}{\Gamma(\alpha j + 1)},$$

$$E_{\alpha,\alpha}(-t^\alpha) = \sum_{j=0}^{\infty} \frac{(-1)^j t^{\alpha j}}{\Gamma(\alpha j + \alpha)}.$$

If  $u_0 > 0$ , then  $u(t) > 0$  as  $E_\alpha(-t^\alpha) > 0$  and  $E_{\alpha,\alpha}(-t^\alpha) > 0$ .

Now, we set the function  $\bar{u}(t) = 1$ ,  $t > 0$ .

As  $0 < u_0 < 1$ , then  $u_0 < \bar{u}(0)$ . In addition, we have

$${}^C D_{0+}^\alpha \bar{u}(t) = 0 = -\bar{u}(t)(1 - \bar{u}(t)).$$

Hence  $\bar{u}$  is an upper solution of the equation (2), and we have  $u(t) < \bar{u}(t) = 1$ , (see [4], Thm. 2.4.3, p. 32).

Now, we examine the large time behavior of the global solution  $0 < u < 1$ .  
For, let us recall the estimates ( see [5] ) :

- For  $0 < \alpha < 1$ , there exists a constant  $c > 0$  such that,

$$0 < E_\alpha(-t^\alpha) \leq \frac{c}{1 + t^\alpha} \leq c, \quad t > 0. \quad (5)$$

- For  $0 < \alpha < 1$ , there exists a constant  $c_1 > 0$  such that

$$0 < t^{\alpha-1} E_{\alpha,\alpha}(-t^\alpha) \leq c_1 t^{\alpha-1}, \quad t > 0. \quad (6)$$

From (4) and using the inequalities (5) and (6), we obtain

$$u(t) \leq cu_0 + c_1 \int_0^t (t-s)^{\alpha-1} u^2(s) ds. \quad (7)$$

We apply Lemma 2.1 to (7) with  $\omega(x) = x^2$ ,  $K(s) = (t-s)^{\alpha-1}$ ,  $A(t) = cu_0$ ,  $B(t) = c_1$ .  
For  $t > T_0$ , we have

$$H(cu_0) + \frac{c_1}{\alpha} t^\alpha \in D(H^{-1}),$$

where  $H(v) = \frac{1}{v_0} - \frac{1}{v}$  and  $H^{-1}(z) = \frac{1}{\frac{1}{v_0} - z}$ ,  $z \neq \frac{1}{v_0}$ .

So we obtain,

$$u(t) \leq H^{-1} \left[ H(cu_0) + \frac{c_1}{\alpha} t^\alpha \right].$$

Therefore

$$u(t) \leq \frac{1}{\frac{1}{cu_0} - \frac{c_1}{\alpha} t^\alpha}, \quad t > T_0.$$

**Part 2.** If  $u_0 > 1$ , then the solution blows-up in a finite time.

1. We show that  $u > 1$ . For, let us define the new unknown function  $w = u - 1$ . The function  $w$  satisfies

$$\begin{cases} {}^C D_{0+}^\alpha w(t) &= w(t)(1 + w(t)), \\ w(0) := w_0 &= u_0 - 1. \end{cases} \quad (8)$$

As  $u_0 > 1$ , then  $w_0 > 0$ . Moreover, we have ([2])

$$w(t) = E_\alpha(t^\alpha)w_0 + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}((t-s)^\alpha) w^2(s) ds.$$

Therefore,  $w > 0$ ; hence  $u > 1$ .

2. We prove that  $u$  blows-up in a finite time.

Since we have  $w(t) = u(t) - 1$ , it is seen that if  $u(t) \rightarrow \infty$  as  $t \rightarrow T^*$ , then  $w(t) \rightarrow \infty$  as  $t \rightarrow T^*$  and vice versa. That is  $w$  and  $u$  will have the same blow-up time.

We now must examine the blow-up properties of  $w$ , the solution of problem (8). These are obtained by comparing  $w(t)$  with the solutions of the following problems :

$$\begin{cases} {}^C D_{0+}^\alpha \bar{w}(t) &= \bar{w}^2(t), \\ \bar{w}(0) &= w_0, \end{cases} \quad (9)$$

and

$$\begin{cases} {}^C D_{0+}^\alpha \tilde{w}(t) &= (\tilde{w}(t) + \frac{1}{2})^2, \\ \tilde{w}(0) &= w_0. \end{cases} \quad (10)$$

We see by comparison ([4]) that

$$\bar{w}(t) \leq w(t) \leq \tilde{w}(t), \quad 0 \leq t < \min\{T_{\bar{w}}, T_{\tilde{w}}\}.$$

Following the paper of Kirk, Olmstead and Roberts [6], we may assert that the solution  $\bar{w}$  ( resp.  $\tilde{w}$ ) blows-up in a finite time  $T_{\bar{w}}$  ( resp.  $T_{\tilde{w}}$ ), such that

$$\left( \frac{\Gamma(\alpha + 1)}{4w_0} \right)^{\frac{1}{\alpha}} \leq T_{\bar{w}} \leq \left( \frac{\Gamma(\alpha + 1)}{w_0} \right)^{\frac{1}{\alpha}},$$

and

$$\left( \frac{\Gamma(\alpha + 1)}{4(w_0 + \frac{1}{2})} \right)^{\frac{1}{\alpha}} \leq T_{\tilde{w}} \leq \left( \frac{\Gamma(\alpha + 1)}{w_0 + \frac{1}{2}} \right)^{\frac{1}{\alpha}}.$$

So we have the following estimates

$$T_{\bar{w}} \leq T^* \leq T_{\tilde{w}}.$$

Whereupon

$$\left( \frac{\Gamma(\alpha + 1)}{4(w_0 + \frac{1}{2})} \right)^{\frac{1}{\alpha}} \leq T^* \leq \left( \frac{\Gamma(\alpha + 1)}{w_0} \right)^{\frac{1}{\alpha}}.$$

□

#### 4. Numerical implementation

In this section, we will approximate the solution  $u$  given by (4). For, we need a numerical approximation of the convolution integral; this can be obtained using the convolution quadrature method.

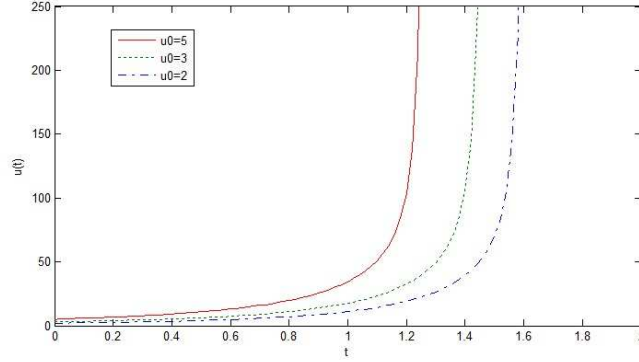


Figure 1: solutions for  $\alpha = 0.5$  and  $u_0 = 5, 3, 2$ .

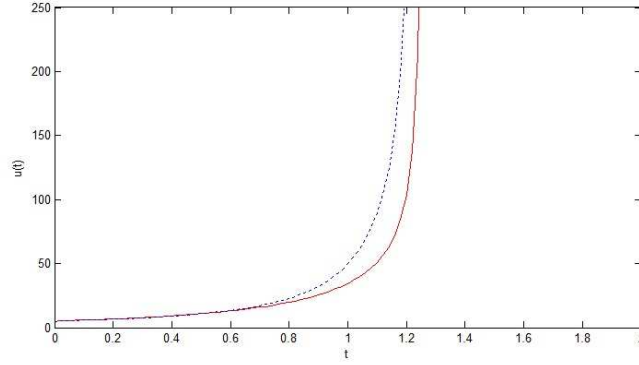


Figure 2: solutions for  $u_0 = 5$  and  $\alpha = 0.3, 0.5$ .

As it has been explained in [7], a convolution quadrature approximates the continuous convolution

$$\int_0^t K(t-s)f(s)ds, \quad t > 0,$$

by a discrete convolution with a step size  $h > 0$ . Then

$$\int_0^{t_n} K(t_n-s)f(s)ds \sim \sum_{j=0}^n \omega_{n-j}f(t_j),$$

where  $t_j = jh$ ,  $j = 0, 1, 2, \dots, n$  and the convolution quadrature weights  $\omega_j$  are determined from their generating power series as

$$\sum_{j=0}^{\infty} \omega_j \zeta^j = \mathcal{L}\left\{K(t) : \frac{\delta(\zeta)}{h}\right\}.$$

Here  $\mathcal{L}\{K(t) : s\}$  is the Laplace transform of  $K(t)$  and  $\delta(\zeta)$  is the generating polynomial for a linear multistep method.

Let  $u_n$  be the approximation of  $u(t_n)$  for  $n \geq 0$ . Using the convolution quadrature method we obtain

$$u_n = (1 - \omega_0)^{-1} \left[ E_\alpha(-t^\alpha)u_0 + \sum_{j=0}^{n-1} \omega_{n-j}u_j \right], \quad n = 1, 2, 3, \dots$$

Now, we introduce the following algorithm which gives the numerical approximation of solution to equation (2).

**Algorithm**

**Input :** Give  $\alpha$ ,  $0 < \alpha < 1$  and  $u_0$ ,  $u_0 > 1$ .

**Initializations :** Discretize the time with a step size  $h > 0$ ;  $t_i = ih$ , for all  $i = 1, 2, \dots, n$ ,  $u_{appx}^1 = u_0$ ,  $u^1 = (u_0)^2$ .

**Step 1 :** Approximate the Mittag-Leffler function **GML**.

**Step 2 :** Calculate convolution quadrature weights **W** using the fast Fourier transform (FFT).

**Step 3 :** Calculate  $u_{appx}^i$ .

**do**

$$u^i = \mathbf{GML} * u_{appx}^1 + \mathbf{W} * u^{i-1}.$$

$$u_{appx}^i = (1 - \mathbf{W}(1))^{-1} * u^i.$$

$$u^i = (u_{appx}^i)^2.$$

$$i = i + 1.$$

**until** ( $u_{appx}^i$  blows up) or ( $i > n$ ).

**Output :** Numerical approximation of  $u$ .

**Example 1.** For Figure1, we set  $\alpha = 0.5$ ; the initial conditions are respectively  $u_0 = 5$ ,  $u_0 = 3$  and  $u_0 = 2$ .

For Figure2, we take the initial condition  $u_0 = 5$  and we plot the solutions; the dotted curve is the solution for  $\alpha = 0.3$  and the solid curve corresponds to the solution for  $\alpha = 0.5$ .

*As it has been proved, the solution blows up in a finite time which depends on  $u_0$  and  $\alpha$ .*

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